Analysis of the out-of-plane Coordinate Transformation to obtain Anisotropic Layered Cloaks

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A possible strategy for avoiding singular material parameters in a transformation-based cloak involves an out-of-plane stretching, calculated to compensate the in-plane singular transformation. In this paper, we use numerical simulations to analyze the relation between the out-of-plane transformation, the resulting material anisotropy and the total scattering cross width. Moreover, because discretization in layers is a common step in a practical design, we also explore its influence, considering a further optimization stage.

***Index Terms*—Cloaking, Coordinate transformation, Electromagnetic metamaterials, Scattering cross section.**

I. Introduction

SINCE the introduction of Transformation Optics (TO) and metamaterials [1], the research in invisibility devices has become one of the most active topics in electromagnetics. Many strategies have been proposed so far, but TO is still a reference of ideal cloaking. It is based on the invariance of Maxwell's Equations under a coordinate transformation, so it can be interpreted as a change in the coordinates or a change in the materials [1]–[3]. The transformation can be analytically described for simple geometries, whilst numerical methods are used for the complex ones. In both cases, however, the designer faces some practical difficulties when implementing the ideal parameters (as revised in II): singularity, continuous non-homogeneity and high anisotropy. In face of it, strategies based on the optimization of a layered cloak became attractive [4]. In many cases, however, TO is still important for having a good start point. Therefore, here we propose a design methodology of a layered cloak with finite properties, which could be used as input in a further optimization process.

II. Methodology

A. Arbitrary Numerical Transformation by Laplace Equation

An arbitrary cloak can be designed by solving the Laplace Equation in the deformed space Ω', establishing the mapping between the original (*virtual*) space Ω and the deformed (*physical*) space Ω' [5]. The solution $\mathbf{x}$ defines the inverse mapping $\mathbf{x} = \mathbf{x}(\mathbf{x}')$, which describes how a given point $\mathbf{x} \in \Omega$ is a function of $\mathbf{x}' \in \Omega'$. As illustrated in Fig. 1, Ω is bounded by $\partial\Omega_+$, whereas Ω' is bounded by $\partial\Omega'_-$ and by $\partial\Omega'_+$. Laplace Equation $\nabla'^2\mathbf{x} = 0$ governs the problem in Ω', subjected to the following boundary conditions: $\mathbf{x} = O$, if $\mathbf{x}' \in \partial\Omega'_-$ and $\mathbf{x} = \mathbf{x}'$, if $\mathbf{x}' \in \partial\Omega'_+$. Note that at this external boundary the unitary mapping guaranties a smooth transition. Conversely, the point $O \in \Omega$ is mapped to the whole internal boundary $\partial\Omega'_-$, which corresponds to the object to be hidden. This singular transformation (an infinitesimal "invisible" point expanded into a finite region) leads to singular material properties.

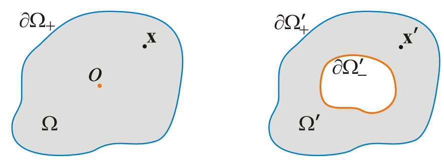

Fig. 1. Arbitrary domains Ω and Ω' involved in the coordinate transformation.

B. Definition of the Material

The partial derivatives of $\mathbf{x}$ with respect to $\mathbf{x}'$ define the Jacobian transformation matrix $\mathbf{A}$ ($A_{ij} = \partial x_i/\partial x'_j$), which is used to calculate the relative permittivity and permeability tensors ($\boldsymbol{\varepsilon}'$ and $\boldsymbol{\mu}'$, respectively), as in (1). They also can be defined by the stretches λ in the principal directions [5], [6]:

$$\boldsymbol{\alpha}' = \frac{\mathbf{A}\boldsymbol{\alpha}\mathbf{A}^{\mathrm{T}}}{|\mathbf{A}|} = \mathrm{diag}\left[\frac{\lambda_1}{\lambda_2\lambda_3}, \frac{\lambda_2}{\lambda_1\lambda_3}, \frac{\lambda_3}{\lambda_1\lambda_2}\right]; \ \boldsymbol{\alpha} = \boldsymbol{\varepsilon},\ \boldsymbol{\mu}. \quad (1)$$

Due to the singular transformation, the stretch λ_2 tends to infinity near $\partial\Omega'_-$, then $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\mu}'$ also become singular. Originally, there is no out-of-plane distortion, that is, $\lambda_3 = 1$. It could be, however, used to compensate the divergence in λ_2 making $\tilde{\lambda}_3 = C_0\left(|x'_1 - x_1| + |x'_2 - x_2|\right)\lambda_2 + 1$, which leads to $\tilde{\boldsymbol{\varepsilon}}'$ and $\tilde{\boldsymbol{\mu}}'$ with only finite values [6]. Note that $\tilde{\lambda}_3 = 1$ if $\mathbf{x} = \mathbf{x}'$ and the constant C_0 controls the out-of-plane stretching in the new non-singular transformation described by $\tilde{\mathbf{A}}$.

C. Anisotropy and a Metric for Evaluating the Material

$\tilde{\mathbf{A}}$ defines how anisotropic $\tilde{\boldsymbol{\varepsilon}}'$ and $\tilde{\boldsymbol{\mu}}'$ are. In order to measure it, we use the anisotropy metric $\tilde{K} = \mathrm{trace}(\tilde{\mathbf{A}}^{\mathrm{T}}\tilde{\mathbf{A}})/3|\tilde{\mathbf{A}}|$. Note that $\tilde{K} \geq 1$ and in fact, it is more appropriated to analyze the maximum anisotropy $\tilde{K}^{\max}$ [7], especially because we are mainly interested in reducing the peak values.

D. Discretization in Homogeneous Layers

The spatially variant tensors (continuous non-homogeneity) are another challenge in realizing transformation-based cloaks.

Then, the material profile is usually approximated by discrete homogeneous layers. Moreover, optimization is often used to achieve a better cloak [4]. In fact, in a typical TO cloak the invisibility is quite affected if the extreme values at the singular region are not well represented [2]. When using a non-singular cloak, however, the finite properties are more suitable for discretization in layers and permit a more proper simulation.

Nevertheless, when dividing in layers other crucial issues emerge, such as: (a) how to define the homogenized material? (b) how many layers? (c) is the impedance well matched between them? (d) is the layered cloak dispersive?

III. Results

Here we assume a z-invariant geometry under a TM$_z$ (H_x, H_y, E_z) incident wave in x^+ direction and evaluate different configurations by $\tilde{K}^{\max}$ and by its total scattering cross width σ^{tot} (always normalized by the uncloaked case, in which $\sigma^{\text{tot}} \equiv 1$). We study the circular cylinder cloak with inner radius a and outer radius b, keeping $a =$ free-space wavelength. For this stated problem, only the in-plane permeability components and the out-of-plane permittivity component become relevant.

A. Cloak with Continuous Non-Homogeneous Material

Figure 2 shows $\tilde{K}^{\max}$ and σ^{tot} for three different b/a ratios and eight values for C_0, including $C_0^\star = 5.61$, that is the value achieved by minimizing $\tilde{K}^{\max}$. Intuitively, the thinner the cloak, the higher $\tilde{K}^{\max}$. Besides, in general lower C_0 achieves lower σ^{tot}, but with higher $\tilde{K}^{\max}$. Higher C_0 is not good at all.

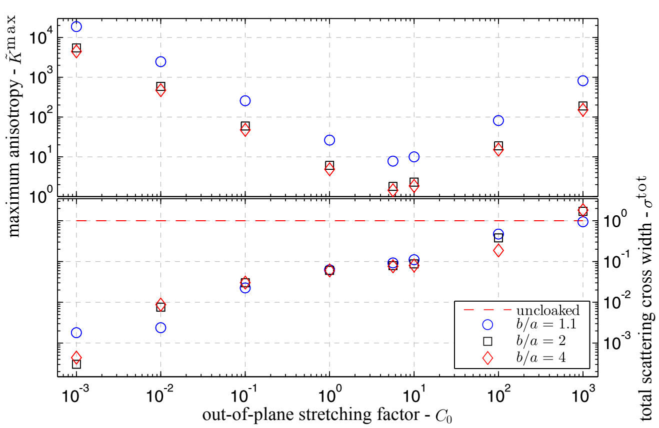

Fig. 2. Maximum anisotropy ($\tilde{K}^{\max}$) and the total scattering cross width (σ^{tot}), depending on the thickness (b/a) and on the stretching factor (C_0).

B. Cloak with Discrete Layers of Homogeneous Material

Now let us get the cloak with $b/a = 2$, divide it in 10 equidistant layers and assume that the homogenized material for a given layer Ω^{layer} would be simply the one with $\tilde{K}^{\max}$ closest to the average $\tilde{K}^{\max}$ in Ω^{layer}. Firstly, we tested that previously calculated $C_0^\star$ in all the layers. Secondly, we calculated a $C_0^{\star\text{layer}}$ for each layer without care about impedance matching. None of these configurations, however, decreased σ^{tot} so much ($\sigma^{\text{tot}} \simeq 0.24$), indicating that the transition to a layered media is not straightforward. In spite of this, based on Fig. 2, we decided to explore the interval $1 < C_0 < 20$ (approximately $\tilde{K}^{\max} < 10$, $\sigma^{\text{tot}} < 0.1$ for the continuous media). When covering this C_0 range in 0.5 steps (see Fig. 3), we found that, for instance, $C_0 = 9.5$ makes $\sigma^{\text{tot}} = 0.086$, $\tilde{K}^{\max} = 2.23$, a quite good preliminary result, considering that no optimization of this layered media was performed.

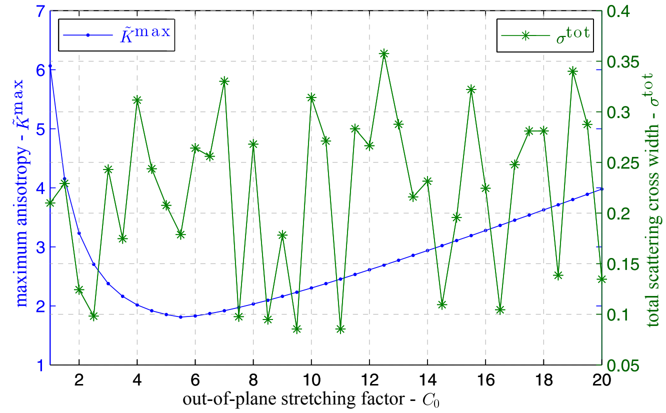

Fig. 3. Maximum anisotropy ($\tilde{K}^{\max}$) and the total scattering cross width (σ^{tot}) as a function of the stretching factor C_0 for the layered cloak.

IV. Conclusion

In the context of designing cloaks with finite parameters, we discussed aspects regarding its discretization after a singularity/anisotropy reduction. We believe that the exposed ideas and results are useful for setting a starting point for an optimization process. Henceforth, at least two strategies can be projected: (a) finding the optimal unique C_0 for the layered cloak which minimizes σ^{tot} under certain material restrictions (i.e. maximum $\tilde{K}^{\max}$); (b) using different $C_0^{\star\text{layer}}$ for each layer, but forcing the impedance matching at the interfaces (similarly to the smooth transition between free-space and the outer interface of the cloak, where $\tilde{\lambda}_3 = 1$). We intend to explore these strategies in the extended paper and in future works.

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